

Why the Standard Model

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Abstract

The Standard Model is based on the gauge invariance principle with gauge group $U(1) \times SU(2) \times SU(3)$ and suitable representations for fermions and bosons, which are begging for a conceptual understanding. We propose a purely gravitational explanation: space-time has a fine structure given as a product of a four-dimensional continuum by a finite noncommutative geometry F . The *raison d'être* for F is to correct the K -theoretic dimension from four to ten (modulo eight). We classify the irreducible finite noncommutative geometries of K -theoretic dimension six and show that the dimension (per generation) is a square of an integer k . Under an additional hypothesis of quaternion linearity, the geometry which reproduces the Standard Model is singled out (and one gets $k = 4$) with the correct quantum numbers for all fields. The spectral action applied to the product $M \times F$ delivers the full Standard Model, with neutrino mixing, coupled to gravity, and makes predictions (the number of generations is still an input).

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1. Introduction

The Standard Model is based on the gauge invariance principle with gauge group

$$G = U(1) \times SU(2) \times SU(3)$$

and suitable representations for fermions and bosons. It involves additional scalar fields, the Higgs fields and a number of key mechanisms such as $V-A$, spontaneous symmetry breaking etc... While the values of the hypercharges can be inferred from the condition of cancelation of anomalies, there is no conceptual reason so far for the choice of the gauge group G as well as for the various representations involved in the construction of SM. Thus under that light the Standard Model appears as one of a plethora of possible quantum field theories, and then needs to be minimally coupled to Einstein gravity.

Our goal in this paper is to show that, in fact, the Standard Model minimally coupled with Einstein gravity appears naturally as pure gravity on a space $M \times F$, where the finite geometry F is one of the simplest and most natural finite noncommutative geometries of the right dimension (6 modulo 8) to solve the fermion doubling problem.

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Such a geometry is given by the following data:

- A finite-dimensional Hilbert space \mathcal{H}
- An antilinear isometry J of \mathcal{H} with $J^2 = \epsilon$
- An involutive algebra \mathcal{A} (over \mathbb{R}) acting in \mathcal{H} , which fulfills the order zero condition

$$[a, b^0] = 0, \quad \forall a, b \in \mathcal{A}, \quad b^0 = Jb^*J^{-1}. \tag{1.1}$$

- A $\mathbb{Z}/2$ -grading γ of \mathcal{H} , such that $J\gamma = \epsilon''\gamma J$
- A self-adjoint operator D in \mathcal{H} such that $JD = \epsilon'DJ$.

In this paper we take the commutation relations i.e. the values of $(\epsilon, \epsilon', \epsilon'') \in \{\pm 1\}^3$ to be specific of K -theoretic dimension 6 modulo 8 i.e. $(\epsilon, \epsilon', \epsilon'') = (1, 1, -1)$. The reason for this choice is that the product geometry $M \times F$ is then of K -theoretic dimension 10 modulo 8 which allows one to use the antisymmetric bilinear form $\langle J\xi, D\eta \rangle$ (for $\xi, \eta \in \mathcal{H}, \gamma\xi = \xi, \gamma\eta = \eta$) to define the fermionic action, so that the functional integral over fermions delivers a Pfaffian rather than a determinant. In other words the “raison d’être” for crossing by F is to shift the K -theoretic dimension from 4 to 10 (modulo 8).

From the mathematical standpoint our road to F is through the following steps

- (1) We classify the irreducible triplets $(\mathcal{A}, \mathcal{H}, J)$.
- (2) We study the $\mathbb{Z}/2$ -gradings γ on \mathcal{H} .
- (3) We classify the subalgebras $\mathcal{A}_F \subset \mathcal{A}$ which allow for an operator D that does not commute with the center of \mathcal{A} but fulfills the “order one” condition:

$$[[D, a], b^0] = 0 \quad \forall a, b \in \mathcal{A}_F. \tag{1.2}$$

The classification in the first step shows that the solutions fall in two classes.

In the first case, the solution is given by an integer k and a real form of the algebra $M_k(\mathbb{C})$. The representation is given by the action by left multiplication on $\mathcal{H} = M_k(\mathbb{C})$, and the isometry J is given by $x \in M_k(\mathbb{C}) \mapsto J(x) = x^*$. There are three real forms: unitary: $M_k(\mathbb{C})$, orthogonal: $M_k(\mathbb{R})$, symplectic: $M_a(\mathbb{H})$ where \mathbb{H} is the skew field of quaternions, and $2a = k$.

In the second case, the algebra is a real form of the sum $M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ of two copies of $M_k(\mathbb{C})$ and while the action is still given by left multiplication on $\mathcal{H} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$, the operator J is given by $J(x, y) = (y^*, x^*)$.

The study (2) of the $\mathbb{Z}/2$ -grading shows that the commutation relation $J\gamma = -\gamma J$ excludes the first case. We are thus left only with the second case and, after considering the grading we are left with the symplectic–unitary algebra: $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. At a more invariant level the Hilbert space is then of the form $\mathcal{H} = \text{Hom}_{\mathbb{C}}(V, W) \oplus \text{Hom}_{\mathbb{C}}(W, V)$ where V is a four-dimensional complex vector space, and W a two-dimensional graded right vector space over \mathbb{H} . The left action of $\mathcal{A} = \text{End}_{\mathbb{H}}(W) \oplus \text{End}_{\mathbb{C}}(V)$ is by composition and its grading as well as the grading of \mathcal{H} comes from the grading of W .

Our main result then is that there exists up to isomorphism a unique involutive subalgebra of maximal dimension \mathcal{A}_F of \mathcal{A}^{ev} , the even part¹ of the algebra \mathcal{A} , which solves (3). This involutive algebra \mathcal{A}_F is isomorphic to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and together with its representation in (\mathcal{H}, J, γ) gives the noncommutative geometry F of [3].

We can then rely on the results of [3], which show that (after the introduction of the multiplicity 3 as the number of generations) the spectral action applied to the inner fluctuations on the product $M \times F$ delivers the Standard Model minimally coupled to gravity. We refer to [3] for the predictions which follow using the spectral action at unification scale.

2. The order zero condition and irreducible pairs (\mathcal{A}, J)

We start with a finite-dimensional Hilbert space \mathcal{H} endowed with an antiunitary operator J such that $J^2 = 1$. For any operator x in \mathcal{H} we let,

$$x^0 = Jx^*J^{-1}. \tag{2.1}$$

¹ One restricts to the even part to obtain an ungraded algebra.

We look for involutive algebras \mathcal{A} of operators in \mathcal{H} such that (cf. (1.1)),

$$[x, y^0] = 0, \quad \forall x, y \in \mathcal{A} \tag{2.2}$$

and that the following two conditions hold:

- (1) The action of \mathcal{A} has a separating vector.²
- (2) The representation of \mathcal{A} and J in \mathcal{H} is irreducible.

The role of the first condition is to abstract a natural property of the action of an algebra of (smooth) functions on the sections of a vector bundle.

The meaning of the second condition is that one cannot find a nontrivial projection $e \in \mathcal{L}(\mathcal{H})$ which commutes with \mathcal{A} and J .

Lemma 2.1. *Assume conditions (2.2) and (1), (2), then, For any projection $e \neq 1$ in the center $Z(\mathcal{A})$ of \mathcal{A} , one has*

$$eJeJ^{-1} = 0. \tag{2.3}$$

For any projections e_j in $Z(\mathcal{A})$ such that $e_1e_2 = 0$ one has

$$e_1Je_2J^{-1} + e_2Je_1J^{-1} \in \{0, 1\}. \tag{2.4}$$

Proof. Let us show (2.3). The projection $eJeJ^{-1}$ commutes with \mathcal{A} and J since JeJ^{-1} commutes with \mathcal{A} by (2.2), and $J(eJeJ^{-1}) = J(JeJ^{-1}e) = eJ^{-1}e = eJe = (eJeJ^{-1})J$. Thus by irreducibility the projection $eJeJ^{-1}$ is equal to 0 or 1 but the latter contradicts $e \neq 1$ since the range of $eJeJ^{-1}$ is contained in the range of e .

Let us show (2.4). Since $e_1e_2 = 0$ the sum $e_1Je_2J^{-1} + e_2Je_1J^{-1}$ is a projection and by the above argument it commutes with \mathcal{A} and J . Thus by irreducibility it is equal to 0 or to 1. \square

We let $\mathcal{A}_{\mathbb{C}}$ be the complex linear space generated by \mathcal{A} in the algebra $\mathcal{L}(\mathcal{H})$ of all operators in \mathcal{H} . It is an involutive complex subalgebra of $\mathcal{L}(\mathcal{H})$ and conditions (2.2), (1) and (2) are still fulfilled.

Lemma 2.2. *Assume conditions (2.2) and (1), (2), then one of the following cases holds*

- The center $Z(\mathcal{A}_{\mathbb{C}})$ is reduced to \mathbb{C} .
- One has $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$ and $Je_1J^{-1} = e_2$ where $e_j \in Z(\mathcal{A}_{\mathbb{C}})$ are the minimal projections of $Z(\mathcal{A}_{\mathbb{C}})$.

Proof. Let us assume that the center $Z(\mathcal{A}_{\mathbb{C}})$ is not reduced to \mathbb{C} . It then contains a partition of unity in minimal projections e_j with $\sum e_j = 1$. By (2.3) we get

$$\sum_{i \neq j} e_iJe_jJ^{-1} = 1.$$

The $e_iJe_jJ^{-1}$ are pairwise orthogonal projections, thus by (2.4) there is a unique pair of indices $\{i, j\} = \{1, 2\}$ such that

$$e_1Je_2J^{-1} + e_2Je_1J^{-1} = 1, \tag{2.5}$$

while the same expression vanishes for any other pair. For $i \notin \{1, 2\}$, one has $e_i = \sum e_iJe_kJ^{-1} = 0$. It follows that all other e_i are zero and thus $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$. Moreover since $e_1 + e_2 = 1$, (2.5) shows that $Je_2J^{-1} \geq e_1$ and $Je_1J^{-1} \geq e_2$ thus $Je_1J^{-1} = e_2$ and $Je_2J^{-1} = e_1$. \square

Remark 2.3. Note that the above statements apply equally well in case $J^2 = \epsilon \in \{\pm 1\}$.

Thus the classification of irreducible pairs splits in the two cases of Lemma 2.2.

² i.e. $\exists \xi \in \mathcal{H}$ such that $\mathcal{A}'\xi = \mathcal{H}$ where \mathcal{A}' is the commutant of \mathcal{A} .

2.1. The case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$

We assume $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$. Then (cf. [5]) there exists $k \in \mathbb{N}$ such that $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ as an involutive algebra over \mathbb{C} . Moreover, the algebra homomorphism

$$\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0 \rightarrow \mathcal{L}(\mathcal{H}), \quad \beta(x \otimes y) = xy^0, \quad \forall x, y \in \mathcal{A}_{\mathbb{C}}, \tag{2.6}$$

is injective since $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0 \sim M_{k^2}(\mathbb{C})$ is a simple algebra.

Lemma 2.4. *The representation β of $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0$ in \mathcal{H} of (2.6) is irreducible.*

Proof. Since $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0 \sim M_{k^2}(\mathbb{C})$, the representation β is a multiple of the unique representation given by the left and right action of $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ on itself. We need to show that the multiplicity m is equal to 1. We let e be a minimal projection of $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$, and let $E = eJeJ^{-1}$. By construction, E is a minimal projection of $\mathcal{B} = \mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0 \sim M_{k^2}(\mathbb{C})$ and thus its range has dimension m . Moreover, by construction, E commutes with J so that J restricts to an antilinear isometric involution of square 1 on $E\mathcal{H}$. Thus $E\mathcal{H}$ is the complexification of a real Hilbert space and J the corresponding complex conjugation. Hence the algebra of endomorphisms of $E\mathcal{H}$ which commute with J is $M_m(\mathbb{R})$ and,³ if $\dim E\mathcal{H} > 1$, it contains a nontrivial idempotent F . For any $\xi \in F\mathcal{H}$, $\eta \in (E - F)\mathcal{H}$ and $b \in \mathcal{B}$ one has

$$\langle b\xi, \eta \rangle = 0 \tag{2.7}$$

since, as E is a minimal projection of \mathcal{B} one has $EbE = \lambda E$ for some $\lambda \in \mathbb{C}$, and $\langle b\xi, \eta \rangle = \langle Ebe\xi, \eta \rangle = 0$. Thus \mathcal{BFH} is a nontrivial subspace which is invariant under \mathcal{B} and J since $J\mathcal{B}J^{-1} = \mathcal{B}$ and J commutes with F . This contradicts the irreducibility condition (2). \square

Proposition 2.5. *Let \mathcal{H} be a Hilbert space of dimension n . Then an irreducible solution with $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$ exists iff $n = k^2$ is a square. It is given by $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution*

$$J(x) = x^*, \quad \forall x \in M_k(\mathbb{C}). \tag{2.8}$$

Proof. We have $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0 \sim M_{k^2}(\mathbb{C})$ and by Lemma 2.4 the representation β in \mathcal{H} is irreducible. This shows that $n = k^2$ is a square. The action of $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0$ by left and right multiplication on $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ (endowed with the Hilbert–Schmidt norm) is a realization of the unique irreducible representation of $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0 \sim M_{k^2}(\mathbb{C})$. In that realization the canonical antiautomorphism

$$\sigma(a \otimes b^0) = b \otimes a^0 \tag{2.9}$$

is implemented by the involution J_0 , $J_0(x) = x^*$ of (2.8) i.e. one has

$$\sigma(x) = J_0x^*J_0^{-1}, \quad \forall x \in \mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0.$$

Since the same property holds for the involution J of the given pair (\mathcal{A}, J) once transported using the unitary equivalence of the representations, it follows that the ratio $J_0^{-1}J$ commutes with \mathcal{B} and hence is a scalar $\lambda \in \mathbb{C}$ of modulus one by irreducibility of β . Adjusting the unitary equivalence by a square root μ of λ (using $\mu J \mu^{-1} = \mu^2 J$) one can assume that $J = J_0$ which gives the desired uniqueness. \square

This determines $\mathcal{A}_{\mathbb{C}}$ and its representation in (\mathcal{H}, J) and it remains to list the various possibilities for \mathcal{A} . Now \mathcal{A} is an involutive subalgebra of $M_k(\mathbb{C})$ such that $\mathcal{A} + i\mathcal{A} = M_k(\mathbb{C})$. The center $Z(\mathcal{A})$ is contained in $Z(M_k(\mathbb{C})) = \mathbb{C}$. If $Z(\mathcal{A}) = \mathbb{C}$ then $i \in \mathcal{A}$ and $\mathcal{A} = M_k(\mathbb{C})$. Otherwise one has $Z(\mathcal{A}) = \mathbb{R}$, \mathcal{A} is a central simple algebra over \mathbb{R} (the simplicity follows from that of $\mathcal{A} + i\mathcal{A} = M_k(\mathbb{C})$) and $\mathcal{A} \cap i\mathcal{A} = \{0\}$ (since this is an ideal in \mathcal{A}). Thus \mathcal{A} is the fixed-point algebra of the antilinear automorphism α of $M_k(\mathbb{C})$ commuting with the $*$ -operation, given by $\alpha(a + ib) = a - ib$ for $a, b \in \mathcal{A}$. There exists (comparing α with complex conjugation) an antilinear isometry I of

³ This is the only place where we use the hypothesis $J^2 = 1$.

\mathbb{C}^k such that $\alpha(x) = IxI^{-1}$ for all $x \in M_k(\mathbb{C})$. One has $\alpha^2 = 1$ and thus $I^2 \in \{\pm 1\}$ (it is a scalar $\lambda \in \mathbb{C}$ of modulus one and commutes with I). Thus \mathcal{A} is the commutant of I and the only two cases are $I^2 = 1$ which gives matrices $M_k(\mathbb{R})$ over \mathbb{R} and $I^2 = -1$. In the latter case the action of I turns \mathbb{C}^k into a right vector space over the quaternions \mathbb{H} and $k = 2a$, $\mathcal{A} = M_a(\mathbb{H})$ is the algebra of endomorphisms of this vector space over \mathbb{H} . We can thus summarize the three possibilities

- $\mathcal{A} = M_k(\mathbb{C})$ (unitary case)
- $\mathcal{A} = M_k(\mathbb{R})$ (orthogonal case)
- $\mathcal{A} = M_a(\mathbb{H})$, for even $k = 2a$, (symplectic case)

while the representation is by left multiplication on $M_k(\mathbb{C})$ and the antilinear involution J is given by (2.8).

Remark 2.6. Note that in the case $J^2 = -1$, the possibility of multiplicity $m = 2$ arises and the dimension of \mathcal{H} is $2k^2$ in that case.

We shall prove below in Lemma 3.1 that the above case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$ is incompatible with the commutation relation $J\gamma = -\gamma J$ for the grading and hence with the K -theoretic dimension 6. Thus we now concentrate on the second possibility: $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$.

2.2. The case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$

We assume $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$. Then there exists $k_j \in \mathbb{N}$ such that $\mathcal{A}_{\mathbb{C}} = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$ as an involutive algebra over \mathbb{C} . We let e_j be the minimal projections $e_j \in Z(\mathcal{A}_{\mathbb{C}})$ with e_j corresponding to the component $M_{k_j}(\mathbb{C})$.

Lemma 2.7. (1) The representation β of $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0$ in \mathcal{H} of (2.6) is the direct sum of two irreducible representations in the decomposition

$$\mathcal{H} = e_1\mathcal{H} \oplus e_2\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \beta = \beta_1 \oplus \beta_2. \tag{2.10}$$

(2) The representation β_1 (resp. β_2) is the only irreducible representation of the reduced algebra of \mathcal{B} by $e_1 \otimes e_2^0$ (resp. $e_2 \otimes e_1^0$).

(3) The dimension of \mathcal{H}_j is equal to k_1k_2 .

Proof. (1) Let $\mathcal{H}_j = e_j\mathcal{H}$. Since $e_j \in Z(\mathcal{A})$ the action of \mathcal{A} in \mathcal{H} is diagonal in the decomposition (2.10). By Lemma 2.2 one has $Je_jJ^{-1} = e_k, k \neq j$, thus the action of \mathcal{A}^0 is also diagonal in the decomposition (2.10). Thus the representation β decomposes as a direct sum $\beta = \beta_1 \oplus \beta_2$. Moreover by Lemma 2.2 the antilinear involution J interchanges the \mathcal{H}_j . Let F_1 be an invariant subspace for the action β_1 of \mathcal{B} in \mathcal{H}_1 . Then $F_1 \oplus JF_1 \subset \mathcal{H}$ is invariant under both \mathcal{B} and J and thus equal to \mathcal{H} by irreducibility which implies $F_1 = \mathcal{H}_1$.

(2) This follows since in each case one gets an irreducible representation of the reduction of \mathcal{B} by the projections $e_i \otimes e_j^0, i \neq j$, which as an involutive algebra is isomorphic to $M_{k_i}(\mathbb{C}) \otimes M_{k_j}(\mathbb{C}) \sim M_{k_1k_2}(\mathbb{C})$.

(3) This follows from (2). \square

Proposition 2.8. Let \mathcal{H} be a Hilbert space of dimension n . Then an irreducible solution with $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$ exists iff $n = 2k^2$ is twice a square. It is given by $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution

$$J(x, y) = (y^*, x^*), \quad \forall x, y \in M_k(\mathbb{C}). \tag{2.11}$$

Proof. Let us first show that $k_1 = k_2$. The dimension of $\mathcal{A}_{\mathbb{C}}$ is $k_1^2 + k_2^2$. The dimension of \mathcal{H} is $2k_1k_2$ by Lemma 2.7. The separating condition implies $\dim \mathcal{A}_{\mathbb{C}} \leq \dim \mathcal{H}$ because of the injectivity of the map $a \in \mathcal{A}_{\mathbb{C}} \mapsto a\xi \in \mathcal{H}$ for ξ such that $\mathcal{A}'\xi = \mathcal{H}$. This gives

$$k_1^2 + k_2^2 \leq 2k_1k_2$$

which is possible only if $k_1 = k_2$. In particular $n = 2k^2$ is twice a square. We have shown that $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ and moreover by Lemma 2.7 the representation β is the direct sum of the irreducible representations of the reduced algebras of \mathcal{B} by the projections $e_1 \otimes e_2^0$ and $e_2 \otimes e_1^0$. Thus we can assume that the representation β of \mathcal{B} is the same as in

the model of Proposition 2.8. It remains to determine the antilinear isometry J . We let $J_0(x, y) = (y^*, x^*)$, $\forall x, y \in M_k(\mathbb{C})$ as in (2.11) and compare the antilinear isometry J of the given pair with J_0 . By the argument of the proof of Proposition 2.5, we get that the ratio $J_0^{-1}J$ commutes with \mathcal{B} and hence is a diagonal matrix of scalars $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. The condition $J^2 = 1$ shows that $\lambda_1 = \lambda_2$. Thus $J = \lambda J_0$ and the argument of the proof of Proposition 2.5 applies to give the required uniqueness. \square

Remark 2.9. One can describe the above solutions (i.e. the algebra $\mathcal{A}_{\mathbb{C}}$ and its representation in \mathcal{H}, J) in a more intrinsic manner as follows. We let V and W be k -dimensional complex Hilbert spaces. Then

$$\mathcal{A}_{\mathbb{C}} = \text{End}_{\mathbb{C}}(W) \oplus \text{End}_{\mathbb{C}}(V). \tag{2.12}$$

We let \mathcal{H} be the bimodule over $\mathcal{A}_{\mathbb{C}}$ given by

$$\mathcal{H} = \mathcal{E} \oplus \mathcal{E}^*, \quad J(\xi, \eta) = (\eta^*, \xi^*) \tag{2.13}$$

where,

$$\mathcal{E} = \text{Hom}_{\mathbb{C}}(V, W), \quad \mathcal{E}^* = \text{Hom}_{\mathbb{C}}(W, V) \tag{2.14}$$

and the algebra acts on the left by composition:

$$(w, v)(g, h) = (w \circ g, v \circ h), \quad \forall (w, v) \in \mathcal{A}_{\mathbb{C}}, (g, h) \in \mathcal{E} \oplus \mathcal{E}^*. \tag{2.15}$$

The various real forms can then be described using additional antilinear isometries of V and W .

3. $\mathbb{Z}/2$ -grading

In the set-up of spectral triples one assumes that in the even case the Hilbert space is $\mathbb{Z}/2$ -graded i.e. endowed with a grading operator γ , $\gamma^2 = 1$, $\gamma = \gamma^*$. This grading should be compatible with a $\mathbb{Z}/2$ -grading of the algebra \mathcal{A} which amounts to asking that $\gamma \mathcal{A} \gamma^{-1} = \mathcal{A}$. One then has $[\gamma, a] = 0$ for any $a \in \mathcal{A}^{\text{ev}}$ the even part of \mathcal{A} .

Lemma 3.1. *In the case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$ of Proposition 2.5, let γ be a $\mathbb{Z}/2$ -grading of \mathcal{H} such that $\gamma \mathcal{A} \gamma^{-1} = \mathcal{A}$ and $J\gamma = \epsilon'' \gamma J$ for $\epsilon'' = \pm 1$. Then $\epsilon'' = 1$.*

Proof. We can assume that $(\mathcal{A}_{\mathbb{C}}, \mathcal{H}, J)$ are as in Proposition 2.5. Let $\delta \in \text{Aut}(\mathcal{A}_{\mathbb{C}})$ be the automorphism given by $\delta(a) = \gamma a \gamma^{-1}$, $\forall a \in \mathcal{A}_{\mathbb{C}}$. One has $\delta^2 = 1$. Similarly one gets an automorphism δ^0 of $\mathcal{A}_{\mathbb{C}}^0$ such that $\delta^0(b^0) = \gamma b^0 \gamma^{-1}$ since $\gamma \mathcal{A}_{\mathbb{C}}^0 \gamma^{-1} = \mathcal{A}_{\mathbb{C}}^0$ using the relation $J\gamma = \epsilon'' \gamma J$. Then $\delta \otimes \delta^0$ defines an automorphism of $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0$ such that

$$\gamma \beta(x) \gamma^{-1} = \beta(\delta \otimes \delta^0(x)), \quad \forall x \in \mathcal{B} = \mathcal{A}_{\mathbb{C}} \otimes \mathcal{A}_{\mathbb{C}}^0.$$

Thus γ implements the tensor product of two automorphisms of $M_k(\mathbb{C})$. These automorphisms are inner and it follows that there are unitary matrices $u, v \in M_k(\mathbb{C})$ such that $\gamma(a) = uav^*$, $\forall a \in \mathcal{A}_{\mathbb{C}}$. One then has

$$J\gamma J^{-1}(a) = (ua^*v^*)^* = vau^*, \quad \forall a \in M_k(\mathbb{C}).$$

Thus the equality $J\gamma J^{-1} = -\gamma$ means that

$$vau^* = -uav^*, \quad \forall a \in M_k(\mathbb{C}),$$

i.e. that $u^*v = z$ fulfills $zaz = -a$ for all a . Thus $z^2 = -1$ and $za = az$ for all a so that $z = \eta i$ for some $\eta \in \{\pm 1\}$. We thus get $v = \eta i u$. Then $\gamma(a) = -\eta i u a u^*$ and $\gamma^2(a) = -u^2 a u^{-2}$. But since $\gamma^2 = 1$ one gets that $a = -u^2 a u^{-2}$ for all $a \in M_k(\mathbb{C})$ which is a contradiction for $a = 1$. \square

Thus Lemma 3.1 shows that we cannot obtain the required commutation relation $J\gamma = -\gamma J$ in the case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}$.

At this point we are at a cross-road. We know that we are in the case $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$ but we must choose the integer k and the real form \mathcal{A} of $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$.

We make the hypothesis that both the grading and the real form come by assuming that the vector space W of Remark 2.9 is a right vector space over \mathbb{H} and is nontrivially $\mathbb{Z}/2$ -graded. The right action of quaternions amounts to

give an antilinear isometry I of W with $I^2 = -1$ (cf. [6] Chapter 3). Since W is a nontrivially $\mathbb{Z}/2$ -graded vector space over \mathbb{H} its dimension must be at least 2 (and hence 4 when viewed as a complex vector space). We choose the simplest case i.e. W is a two-dimensional space over \mathbb{H} , and there is no ambiguity since all nontrivial $\mathbb{Z}/2$ -gradings are equivalent. A conceptual description of the algebra \mathcal{A} and its representation in \mathcal{H} is then obtained from Remark 2.9. We let V be a four-dimensional complex vector space. Our algebra is

$$\mathcal{A} = \text{End}_{\mathbb{H}}(W) \oplus \text{End}_{\mathbb{C}}(V) \sim M_2(\mathbb{H}) \oplus M_4(\mathbb{C}). \tag{3.1}$$

It follows from the grading of W that the algebra (3.1) is also $\mathbb{Z}/2$ -graded, with nontrivial grading only on the $M_2(\mathbb{H})$ -component. We still denote by γ the gradings of $\mathcal{E} = \text{Hom}_{\mathbb{C}}(V, W)$ and $\mathcal{E}^* = \text{Hom}_{\mathbb{C}}(W, V)$ given by composition with the grading of W . We then have, with the notations of (2.13),

Proposition 3.2. *There exists up to equivalence a unique $\mathbb{Z}/2$ -grading of \mathcal{H} compatible with the graded representation of \mathcal{A} and such that:*

$$J\gamma = -\gamma J. \tag{3.2}$$

It is given by

$$\mathcal{H} = \mathcal{E} \oplus \mathcal{E}^*, \quad \gamma(\xi, \eta) = (\gamma\xi, -\gamma\eta). \tag{3.3}$$

Proof. By construction, the grading (3.3) is a solution. Given two gradings γ_j fulfilling the required conditions one gets that their ratio $\gamma_1\gamma_2$ commutes with \mathcal{A} (since both define the grading of \mathcal{A} by conjugation) and with J . Thus by irreducibility one gets that $\gamma_1\gamma_2 \in \pm 1$. Changing γ to $-\gamma$ amounts to changing the grading of W to its opposite, but up to isomorphism this gives the same result. \square

Remark 3.3. The space $\mathcal{E} = \text{Hom}_{\mathbb{C}}(V, W)$ is related to the classification of instantons (cf. Equation (1.1) Chapter III of [1]).

4. The subalgebra and the order one condition

We take $(\mathcal{A}, \mathcal{H}, J, \gamma)$ from the above discussion, i.e. (3.1) and Proposition 3.2.

The center of our algebra $Z(\mathcal{A})$ is nontrivial and in that way the corresponding space is not connected. We look for “Dirac operators” D which connect nontrivially the two pieces (we call them “off-diagonal”) i.e. operators such that:

$$[D, Z(\mathcal{A})] \neq \{0\}. \tag{4.1}$$

The main requirement on such operators is the order one condition (1.2). We now look for subalgebras $\mathcal{A}_F \subset \mathcal{A}^{\text{ev}}$, the even part of \mathcal{A} , for which this order condition ($\forall a, b \in \mathcal{A}_F$) allows for operators which fulfill (4.1). We can now state our main result which recovers the main “input” of [3] in a more conceptual manner.

Theorem 4.1. *Up to an automorphism of \mathcal{A}^{ev} , there exists a unique involutive subalgebra $\mathcal{A}_F \subset \mathcal{A}^{\text{ev}}$ of maximal dimension admitting off-diagonal Dirac operators. It is given by*

$$\mathcal{A}_F = \{(\lambda \oplus q, \lambda \oplus m) \mid \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})\} \subset \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C}), \tag{4.2}$$

using a field morphism $\mathbb{C} \rightarrow \mathbb{H}$. The involutive algebra \mathcal{A}_F is isomorphic to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and together with its representation in (\mathcal{H}, J, γ) it gives the noncommutative geometry F of [3].

We now give the argument (which is similar to that of [3] Proposition 2.11). Let us consider the decomposition of (2.10),

$$\mathcal{H} = e_1\mathcal{H} \oplus e_2\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

We consider an involutive subalgebra $\mathcal{A}_F \subset \mathcal{A}^{\text{ev}}$ and let π_j be the restriction to \mathcal{A}_F of the representation of \mathcal{A} in \mathcal{H}_j . We have (cf. [3] Lemma 2.12),

Lemma 4.2. *If the representations π_j are disjoint, then there is no off-diagonal Dirac operator for \mathcal{A}_F .*

Proof. By construction, the projections e_j are the minimal projections in $Z(\mathcal{A})$ and since $Je_1J^{-1} = e_2$, they are also the minimal projections in $Z(\mathcal{A}^0)$. Moreover, one has

$$\pi_j(a) = ae_j = e_ja = e_jae_j, \quad \forall a \in \mathcal{A}_F.$$

Let us assume that the representations π_j are disjoint. For any operator T in \mathcal{H} , one has

$$[T, a] = 0, \quad \forall a \in \mathcal{A}_F \Rightarrow [e_1Te_2, a] = 0, \quad \forall a \in \mathcal{A}_F \Rightarrow e_1Te_2 = 0, \tag{4.3}$$

since any intertwining operator such as e_1Te_2 must vanish as the two representations are disjoint. The same conclusion applies to e_2Te_1 . Similarly one gets, after conjugating by J ,

$$[T, a^0] = 0, \quad \forall a \in \mathcal{A}_F \Rightarrow e_1Te_2 = e_2Te_1 = 0 \tag{4.4}$$

(one has $[JTJ^{-1}, a] = 0$ for all $a \in \mathcal{A}$ hence $e_2JTJ^{-1}e_1 = 0$ and $e_1Te_2 = 0$). Now let the operator D satisfy the order one condition

$$[[D, a], b^0] = 0 \quad \forall a, b \in \mathcal{A}_F. \tag{4.5}$$

It follows from (4.4) that $e_1[D, a]e_2 = 0$ for all $a \in \mathcal{A}_F$. Since the e_j commute with a this gives

$$[e_1De_2, a] = 0, \quad \forall a \in \mathcal{A}_F.$$

By (4.3) we thus get $e_1De_2 = 0$ and there is no off-diagonal Dirac operator for \mathcal{A}_F . \square

For any operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ we let

$$\mathcal{A}(T) = \{b \in \mathcal{A}^{\text{ev}} \mid \pi_2(b)T = T\pi_1(b), \pi_2(b^*)T = T\pi_1(b^*)\}. \tag{4.6}$$

It is by construction an involutive unital subalgebra of \mathcal{A}^{ev} .

We now complete the proof of [Theorem 4.1](#). We let $\mathcal{A}_F \subset \mathcal{A}^{\text{ev}}$ be an involutive subalgebra with an off-diagonal Dirac operator. Then by [Lemma 4.2](#), the representations π_j are not disjoint and thus there exists a nonzero operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $\mathcal{A}_F \subset \mathcal{A}(T)$. If we replace $T \rightarrow c_2Tc_1$ where c_j belongs to the commutant of \mathcal{A}^{ev} , we get

$$\mathcal{A}(T) \subset \mathcal{A}(c_2Tc_1),$$

since the c_j commute with the $\pi_j(b)$. This allows one to assume that the support of T is contained in an irreducible subspace of the restriction of the action of \mathcal{A}^{ev} on \mathcal{H}_1 and that the range of T is contained in an irreducible subspace of the restriction of the action of \mathcal{A}^{ev} on \mathcal{H}_2 . We can thus assume that π_1 is the irreducible representation of one of the two copies of \mathbb{H} in \mathbb{C}^2 , while π_2 is the irreducible representation of $M_4(\mathbb{C})$ in \mathbb{C}^4 . We can remove the other copy of \mathbb{H} and replace $\mathcal{A}^{\text{ev}} = \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ by its projection $\mathcal{C} = \mathbb{H} \oplus M_4(\mathbb{C})$. The operator T is a nonzero operator $T : \mathbb{C}^2 \rightarrow \mathbb{C}^4$ and with

$$\mathcal{C}(T) = \{b \in \mathcal{C} \mid \pi_2(b)T = T\pi_1(b), \pi_2(b^*)T = T\pi_1(b^*)\} \tag{4.7}$$

we have that $\mathcal{A}(T) = \{(q, y) \mid q \in \mathbb{H}, y \in \mathcal{C}(T)\}$. In particular $\dim \mathcal{A}(T) = 4 + \dim \mathcal{C}(T)$. Let us first assume that the rank of T is equal to 2. The range E of T is a two-dimensional subspace of \mathbb{C}^4 and by (4.7) it is invariant under the action of $b \in \mathcal{C}$ as well as its orthogonal complement. This shows that in that case

$$\mathcal{C}(T) \subset \mathbb{H} \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

and moreover the relation (4.7) shows that the component of $\pi_2(b)$ in the copy of $M_2(\mathbb{C})$ corresponding to E is determined by the quaternion component $\pi_1(b)$. Thus we get

$$\mathcal{C}(T) \subset \mathbb{H} \oplus M_2(\mathbb{C}).$$

In particular the dimension fulfills

$$\dim_{\mathbb{R}} \mathcal{C}(T) \leq 4 + 8 = 12.$$

Let us now consider the other possibility, namely that the rank of T is equal to 1. The range E of T is a one-dimensional subspace of \mathbb{C}^4 and by (4.7) it is invariant under the action of $b \in \mathcal{C}$ as well as its orthogonal complement. The support

$S \subset \mathbb{C}^2$ of T is a one-dimensional subspace and since both the unitary group $SU(2)$ of \mathbb{H} and $U(4)$ of $M_4(\mathbb{C})$ act transitively on the one-dimensional subspaces (of \mathbb{C}^2 and \mathbb{C}^4) we are reduced to the case

$$S = \{(a, 0) \in \mathbb{C}^2\}, \quad E = \{(a, 0, 0, 0) \in \mathbb{C}^4\}, \quad T(a, b) = (a, 0, 0, 0), \quad \forall a, b \in \mathbb{C}.$$

One then obtains, for the natural embedding

$$\mathbb{C} \subset \mathbb{H}, \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

that,

$$\mathcal{C}(T) = \{(\lambda, \lambda \oplus m) \in \mathbb{H} \oplus M_4(\mathbb{C}) \mid \lambda \in \mathbb{C}, m \in M_3(\mathbb{C})\}.$$

Thus the dimension fulfills

$$\dim_{\mathbb{R}} \mathcal{C}(T) = 2 + 18 = 20.$$

Thus we see that this gives the solution with maximal dimension, and it is unique up to an automorphism of \mathcal{A}^{ev} .

We can now combine the above discussion with the result of [3] [Theorem 4.3](#) and get,

Theorem 4.3. *Let M be a Riemannian spin 4-manifold and F the finite noncommutative geometry of K -theoretic dimension 6 described above, but with multiplicity⁴ 3. Let $M \times F$ be endowed with the product metric.*

- (1) *The unimodular subgroup of the unitary group acting by the adjoint representation $Ad(u)$ in \mathcal{H} is the group of gauge transformations of SM .*
- (2) *The unimodular inner fluctuations of the metric give the gauge bosons of SM .*
- (3) *The full Standard Model (with neutrino mixing and seesaw mechanism) minimally coupled to Einstein gravity is given in Euclidean form by the action functional*

$$S = \text{Tr}(f(D_A/\Lambda)) + \frac{1}{2} \langle J\tilde{\xi}, D_A\tilde{\xi} \rangle, \quad \tilde{\xi} \in \mathcal{H}_{cl}^+, \tag{4.8}$$

where D_A is the Dirac operator with the unimodular inner fluctuations.

We refer to [3] for the notations and for the predictions.

Remark 4.4. The “unimodularity” condition imposed in [Theorem 4.3](#) (cf. [3]) on our gauge transformations can now be viewed as the restriction to \mathcal{A}_F of the condition giving the group of inner automorphisms of \mathcal{A} . Indeed this group is described as the unimodular unitary group:

$$\text{Int}(\mathcal{A}) \sim SU(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = 1, \det(u) = 1\}. \tag{4.9}$$

This applies also to the product geometry by the manifold M .

5. Conclusion

The fermion doubling problem requires (cf. [2,4]) crossing the ordinary four-dimensional continuum by a space of K -theoretic dimension 6. We have shown in this paper that the classification of the finite noncommutative geometries of K -theoretic dimension 6 singles out the algebras which are real forms of $M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ acting in the Hilbert space of dimension $2k^2$ by left multiplication, together with a specific antilinear isometry. This predicts the number of fermions per generation to be a square and under our *hypothesis* about the role of quaternions the simplest case is with $k = 4$ and gives the noncommutative geometry of the Standard Model in all its details, including the representations of fermions and bosons and the hypercharges.

While we have been able to find a short path to the Standard Model coupled to gravity from simple geometric principles using noncommutative geometry and the spectral action, there are still a few forks along the way where the choice we made would require a better justification. The list is as follows:

⁴ i.e. we just take three copies of \mathcal{H} .

Why \mathbb{H} : The field \mathbb{H} of quaternions plays an important role in our construction, since we assumed that both the grading and the real form come from W being quaternionic. This is begging for a better understanding. The role of quaternions in the classification of instantons [1] is the one possible starting point as well as the role of discrete symmetries (cf. [6]).

Three generations: We took the number $N = 3$ of generations as an input which gave the multiplicity 3. From the physics standpoint, violation of CP is a reason for $N \geq 3$ but it remains to find a convincing mathematical counterpart.

Massless photon: In the classification [3] of the operators D for the finite geometry F , we impose that D commutes with the subalgebra $\{(\lambda, \lambda, 0); \lambda \in \mathbb{C}\}$. While the physics meaning of this condition is clear since it amounts to the masslessness of the photon, a conceptual mathematical reason for only considering metrics fulfilling this requirement is out of sight at the moment.

Our approach delivers the unique representation for the fermions, a property which is only shared with the $SO(10)$ grand-unified theory. One of the main advantages of our approach with respect to unified theories is that the reduction to the Standard Model group $G = U(1) \times SU(2) \times SU(3)$ is not due to a plethora of added scalar Higgs fields, but is naturally imposed by the order one condition.

The spectral action of the Standard Model comes out almost uniquely, predicting the number of fermions, their representations, the gauge group and their quantum numbers as well as the Higgs mechanism, with very little input. This manages to combine the advantages of Kaluza–Klein unification (we are dealing with pure gravity on a space of K -theoretic dimension ten) with those of grand-unification such as $SO(10)$ (including the unification of coupling constants) without introducing unobserved fields and an infinite tower of states.

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